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Goldstone's theorem and related topics

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I Symmetries and conserved currents in field theory

The purpose of these lectures will be to discuss some problems related with the existence of symmetries associated with conserved currents in quantum field theory.

Before entering the quantum case let us briefly review some classical results.

Take a Lagrangean density

$$\mathscr{L}(\Phi_i(x), \partial_\mu \Phi_i(x)) \tag{I.1}$$

which gives us by the principle of minimal action

$$\delta \int \mathscr{L} d^4x = 0 \tag{I.2}$$

the Euler-Lagrange equations of motion

$$\frac{\partial}{\partial x^\mu} \frac{\partial \mathscr{L}}{\partial \partial_\mu \Phi_i} - \frac{\partial \mathscr{L}}{\partial \Phi_i} = 0 \tag{I.3}$$

If the Lagrangean (I.1) is invariant under an n parametric transformation group

$$\Phi_i(x) \to V_{ij}(\lambda_1 \ldots \lambda_n) \Phi_j(x) \tag{I.4a}$$

$$V = e^{-i\lambda_k I^k} \quad (I^k \text{ infinitesimal generators}) \tag{I.4b}$$

we obtain on one hand the invariance of the equations of motion (I.3) under the transformation (I.4) and on the other hand Noether's theorem gives us n conserved currents

$$\mathscr{J}_\mu^k = -i \frac{\partial \mathscr{L}}{\partial_\mu \Phi} I^k \Phi \tag{I.5}$$

(in matrix notation).

Since the momentum canonically conjugate to Φ is

$$\Pi = \frac{\partial \mathscr{L}}{\partial_0 \Phi} \tag{I.6}$$

satisfying the well-known Poisson bracket relation

$$\{\Phi_i(\mathbf{x}), \Pi_j(\mathbf{y})\}_p = \delta_{ij}\delta(\mathbf{x}-\mathbf{y}) \tag{I.7}$$

we have

$$\{\mathscr{J}_0^k(\mathbf{x}), \Phi_i(\mathbf{y})\}_p = iI_{ij}^k \Phi_j(\mathbf{y})\delta(\mathbf{x}-\mathbf{y}) \tag{I.8}$$

or introducing the conserved "charge"

$$Q^k = \int \mathscr{J}_0^k(\mathbf{x}) d^3x \tag{I.9}$$

$$\{Q^k, \Phi_i(y)\}_p = iI_{ij}^k \Phi_j(y) \tag{I.10}$$

Q^k is therefore the generator of the infinitesimal canonical transformations corresponding to (I.4).

In the traditional formulation of quantum field theory one works formally with a Lagrangean (I.1) which leads to the equations of motion for the quantized field in close correspondence with the classical treatment.

Also in a very formal fashion one applies Noether's theorem to obtain conserved currents given essentially by (I.5) which satisfy

$$[\mathscr{J}_0^k(\mathbf{x}), \Phi_i(\mathbf{y})] = -I_{ij}^k \Phi_j(y)\delta(\mathbf{x}-\mathbf{y}) \tag{I.11}$$

Again it is usual to introduce an "operator"

$$U(\lambda) = e^{i\lambda_k Q^k} \tag{I.12}$$

supposedly implementing unitarily the symmetry

$$U(\lambda)\Phi_i(x)U^{-1}(\lambda) = V_{ij}(\lambda)\Phi_j(x) \tag{I.13}$$

It is argued then that the physical vacuum has no quantum numbers and therefore should be left invariant by the symmetry, implying in

$$Q^k|0\rangle = 0 \tag{I.14}$$

All those arguments are however at most of heuristic value for several reasons:

I The Lagrangean, equations of motion and currents involve products of field operators at the same point and therefore are ill defined quantities whose proper meaning should be obtained by limiting procedures starting from different space-time points [1].

2 The construction of the "charge" from the density (I.9) requires in the classical case the hypothesis that the fields should vanish at infinity to ensure convergence of the integral, what is physically very reasonable. In the quantum case the existence of vacuum fluctuations occurring all over space (translation invariance) does not allow us to take the quantum analogue of (I.9) as a well-defined operator even if a meaning has been given to the density. (This is the source of the spontaneous symmetry breakdown.)

In order to avoid those difficulties the more ambitious program would be to show that under certain conditions (for instance existence of a mass gap) any continuous group of local $*$-automorphisms of the operator algebra is unitarily implementable. I think we are far from such a proof. A slightly less ambitious task but also to my knowledge unaccomplished would be to study the class of automorphisms which lead to conserved currents obtaining thus a quantum Noether theorem.

Finally one can by-pass the harder part by assuming the existence of a conserved current defined as an operator valued distribution and concentrate one's attention to difficulty nb. 2. This will be our purpose in those lectures. We shall concentrate mainly on internal symmetries of the kind (I.4) which do not change the space-time coordinate. Many of our conclusions are however generalizable to a larger class of symmetry groups.

We shall work with quantized fields $\Phi_i(x)$ satisfying Wightman's axioms [2]. It would also be possible to base our discussion on the Haag-Kastler [3] algebraic framework.

From the $\Phi_i(x)$ the basic quantized fields of the theory we go over to the (quasi-local) Wightman polynomials [2]

$$\mathscr{P} = \sum_{n=1}^{n} \int \sum_{i_1 \dots i_n} f_n \quad (x_1 \dots x_n) \Phi_{i_1}(x_1) \dots \Phi_{i_n}(x_n) d^4x_1 \dots d^4x_n \quad \text{(I.15)}$$

with the f_n S class functions.

Particularly important will be the local Wightman polynomials associated to a finite space-time region θ

$$A = \sum_{n=1}^{n} \sum \int h_{n\,i_1 \ldots i_n} (x_1 \ldots x_n) \Phi_{i_1}(x_1) \ldots \Phi_{i_n}(x_n) d^4x_1 \ldots d^4x_n$$

with h_n out of D and having support in θ.

At its most basic level a symmetry of a Q.F.T. is a correspondence

$$A \to A_\lambda \tag{I.17}$$

induced by (I.4) that leaves invariant the equations of motion and commutation relations of the theory, i.e., that preserves its algebraic structure (automorphism of the operator algebra). [4].

From the point of view of observable consequences, that is, to obtain from the symmetry relations between cross-sections, multiplet-structure, etc. ... it is necessary to have the correspondence (I.17) unitarily implemented with a $U(\lambda_1 \ldots \lambda_n) = U(\lambda)$ such that

$$U(\lambda) A U^{-1}(\lambda) = A_\lambda \tag{I.18}$$

$$U(\lambda)|0\rangle = |0\rangle \tag{I.19}$$

We *assume* the existence of a conserved current $\mathscr{J}^\mu(x)$ with $\mathscr{J}^\mu$ a hermitian and local field

$$\partial_\mu \mathscr{J}^\mu(x) = 0 \tag{I.20}$$

and

$$\left.\frac{dA_\lambda}{d\lambda}\right|_{\lambda=0} = \underset{R>R_0}{i[\mathscr{J}^0(f_d f_R), A]} \tag{I.21}$$

with

$$(\mathscr{J}^0 f_d f_R) = \int \mathscr{J}^0(x) f_d(x_0) f_R(\mathbf{x}) d^4x \tag{I.22}$$

and f_d, f_R S class functions with

$$\begin{aligned} f_R(\mathbf{x}) &= 1 \qquad |\mathbf{x}| < R \\ f_R(\mathbf{x}) &= 0 \qquad |\mathbf{x}| > R + \varepsilon \end{aligned} \tag{I.23}$$

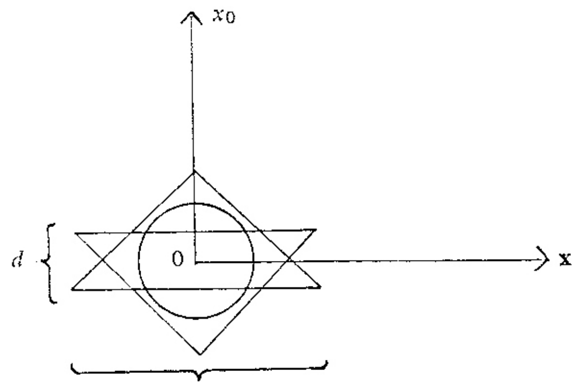
$$\begin{aligned} f_d(x_0) &= 0 \qquad |x_0| > d \\ \int f_d(x_0) dx_0 &= 1 \end{aligned} \tag{I.24}$$

and R_0 being such that the points

$$(x_0, \mathbf{x}) \text{ with } \quad \begin{array}{l} |x_0| < d \\ |\mathbf{x}| > R_0 \end{array}$$

lie outside of the light-cone of the region θ to which A is associated.

FIG. 1



Equation (I.21) is a careful way of expressing the content of the equal-time commutation relation (I.11) which is thus believed to be basically correct. (Since (I.11) follows by blindly applying equal-time canonical commutation relations, I think it is worthwhile to check its validity in perturbation theory [5], specially since there are known examples of soluble field theories in 2 dimensional space-time where one obtains results different from the canonical ones [6]).

Before proceeding any further consistency requires that the r.h.s. of equation (I.21) be shown independent of $f_d f_R$ for $R > R_0$.

The independence with f_R is a trivial consequence of local commutativity since

$$[(\mathcal{J}^0 f_d f^{(1)}_{R_1}), A] - [\mathcal{J}^0(f_d f^{(2)}_{R_2}), A] = 0 \qquad \text{(I.25)}$$
$$R_1 > R_0 \qquad\qquad R_2 > R_0$$

because $f^{(1)}_{R_1}(\mathbf{x}) = f^{(2)}_{R_2}(\mathbf{x}) = 1$ for $|\mathbf{x}| < R_0$ from (I.23) and those are the only points that contribute to the commutator since

$$[\mathcal{J}^0(x), A] = 0 \text{ for } |x_0| < d \; |\mathbf{x}| > R_0 \qquad \text{(I.26)}$$

The independence with f_d is shown taking

$$\hat{f}(x_0) = \int_{-\infty}^{x_0} (f^1_{d_1}(x_0') - f^{(2)}_{d_2}(x_0'))dx_0' \tag{I.27}$$

where without loss of generality we take $d_1 > d_2$.

From (I.24) we see that $\hat{f}(x_0)$ vanishes outside an interval of thickness d_1 around the origin and since

$$\frac{d}{dx_0}\hat{f}(x_0) = f^1_{d_1}(x_0) - f^2_{d_2}(x_0) \tag{I.28}$$

one gets using the conservation law (I.20)

$$\mathscr{I}^0\left(\frac{d\hat{f}}{dx_0} f_R\right) = \mathscr{I}^0(f^1_{d_1} f_R) - \mathscr{I}^0(f^2_{d_2} f_R) = \mathscr{I}(\hat{f}\nabla f_R) \tag{I.29}$$

with

$$\mathscr{I}(\hat{f}\nabla f_R) = \int \mathscr{I}(x_0\mathbf{x})\hat{f}(x_0)\nabla f_R(\mathbf{x})d^4x \tag{I.30}$$

Since $\nabla f_R(\mathbf{x}) = 0$ for $|\mathbf{x}| < R$ from (I.23) one gets again by local commutativity

$$\underset{R>R_0}{[\mathscr{I}(\hat{f}\nabla f_R), A]} = 0 \tag{I.31}$$

and therefore

$$\underset{R>R_0}{[\mathscr{I}^0(f^1_{d_1} f_R), A]} = \underset{R>R_0}{[\mathscr{I}^0(f^2_{d_2} f_R), A]} \qquad \text{q.e.d.} \tag{I.32}$$

The charge operator will be now defined on the dense set of states obtained by applying local polynomials on the vacuum by

$$QA|0\rangle \overset{\text{def}}{=} \underset{R>R_0}{[\mathscr{I}^0 f_d f_R, A]}|0\rangle = \left.\frac{dA_\lambda}{id\lambda}\right|_{\lambda=0}|0\rangle \tag{I.33}$$

In order that this be a consistent definition of a hermitian operator it will be necessary to demonstrate a theorem known as the Goldstone theorem [7, 8].

II Goldstone's theorem

Th. If there are no zero mass particles in the theory, then

$$\langle 0|\underset{R>R_0}{[\mathscr{I}^0(f_d, f_R), A]}|0\rangle = 0 \tag{II.1}$$

The independence with f_d is shown taking

$$\hat{f}(x_0) = \int_{-\infty}^{x_0} (f_{d_1}^1(x_0') - f_{d_2}^{(2)}(x_0'))dx_0' \tag{I.27}$$

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II Goldstone's theorem

Th. If there are no zero mass particles in the theory, then

$$\langle 0|\underset{R>R_0}{[\mathscr{J}^0(f_d, f_R), A]}|0\rangle = 0 \tag{II.1}$$

Proof Take the v.e.v. of the commutator of the charge density with A, that satisfies a generalization of the Källen–Lehmann representation [9] due to Araki, Hepp and Ruelle [10]

$$\langle 0|[\mathscr{J}^0(x), A]|0\rangle = \int_0^\infty d\mu^2 \int d^3y \Delta(\mathbf{x}-\mathbf{y}, x_0, \mu^2)\rho_1(\mu^2, \mathbf{y}) + \\ + \int_0^\infty d\mu^2 \int d^3y \frac{\partial}{\partial x^0} \Delta(\mathbf{x}-\mathbf{y}, x_0, \mu^2)\rho_2(\mu^2, \mathbf{y}) \qquad \text{(II.2)}$$

where $\Delta(\mathbf{x}, x_0, \mu^2)$ is the Pauli-Jordan function for a scalar field of mass μ and ρ_1, ρ_2 are measures in μ^2 having compact support in $\mathbf{y}$ as a consequence of local commutativity.

One can write $(i = 1, 2)$

$$\rho_i(\mu^2, \mathbf{y}) = \bar{\rho}_i(\mu^2)\delta^3(\mathbf{y}) + \nabla \cdot \mathfrak{G}_i(\mu^2, \mathbf{y}) \qquad \text{(II.3)}$$

with $\mathfrak{G}_i(\mu^2, \mathbf{y})$ also of compact support in $\mathbf{y}$.

To verify (II.3) observe that

$$\mathfrak{G}_i^{(1)}(\mu^2, \mathbf{y}) = \int_{-\infty}^{y_1} \{\rho_i(\mu^2, y_1', y_2 y_3) - \delta(y_1')\int_{-\infty}^{+\infty} \rho_i(\mu^2, y_1'', y_2, y_3) dy_1''\} dy_1' \qquad \text{(II.4)}$$

has compact support and

$$\rho_i(\mu^2, \mathbf{y}) = \delta(y_1) \int_{-\infty}^{+\infty} \rho(\mu^2, y_1', y_2, y_3) dy_1' + \frac{\partial \mathfrak{G}_i^{(1)}}{\partial y_1} \qquad \text{(II.5)}$$

Repeating the argument above for the coefficient of $\delta(y_1)$ in (II.5) with respect to the variables y_2, y_3 one arrives at (II.3). Using (II.2) and (II.3) we get

$$\langle 0|[\mathscr{J}^0(x_0, f_R), A]|0\rangle_{R > R_0(x_0)} = \int_0^\infty d\mu^2 \int f_R(\mathbf{x}) d^3x \{\bar{\rho}_1(\mu^2)\Delta(\mathbf{x}, x_0 \mu^2) + \\ + \bar{\rho}_2(\mu^2) \frac{\partial}{\partial x_0} \Delta(\mathbf{x}, x_0, \mu^2) \qquad \text{(II.6)}$$

where to simplify matters we have taken $f_d(x_0') = \delta(x_0 - x_0')$ and the terms with $\nabla \mathfrak{G}$ vanish using the fact that $\Delta(x^2, \mu^2)$ is zero for $x^2 < 0$ and $f_R(\mathbf{x}) = 1$ for $|\mathbf{x}| < R$.

On the other hand from (I.28–32)

$$\frac{d}{dx_0}\langle 0|[\mathscr{J}^0(x_0, f_R), A]|0\rangle_{R>R_0(x_0)} = 0 \tag{II.7}$$

With (II.6) and (II.7) one gets

$$\int_0^\infty \bar{\rho}_1(\mu^2)\cos(\mu x_0)d\mu^2 = 0$$

$$\int_0^\infty \bar{\rho}_2(\mu^2)\mu\sin(\mu x_0)d\mu^2 = 0 \tag{II.8}$$

what implies in

$$\bar{\rho}_1(\mu^2) = 0$$
$$\bar{\rho}_2(\mu^2) = \lambda\delta(\mu^2) \tag{II.9}$$

and therefore

$$\langle 0|[\mathscr{J}^0(x_0, f_R), A]|0\rangle_{R>R_0(x_0)} = 0 \text{ unless } \lambda \neq 0 \tag{II.10}$$

To conclude our proof it is enough to show that $\lambda \neq 0$ implies discrete states of zero mass.

In fact using (II.3, 9) one obtains

$$\int \rho_2(\mu^2, \mathbf{y})V(\mathbf{y})d^3y = \lambda \tag{II.11}$$

with $V(\mathbf{y})$ any sufficiently smooth function which is 1 in the compact region when $\mathfrak{G}_2$ is different from zero and has compact support. Using

$$\Delta(\mathbf{x}, x_0, \mu^2)\Big|_{x_0=0} = 0$$

$$\frac{\partial}{\partial x_0}\Delta(\mathbf{x}, x_0, \mu^2)\Big|_{x_0=0} = \delta^3(\mathbf{x}) \tag{II.12}$$

and recalling that the integration over μ^2 comes from intermediate states of mass μ we see with (II.11, 12, 2) that

$$\langle 0|\int V(\mathbf{x})\mathscr{J}^0(0, \mathbf{x})d^3xE(M^2)A - AE(M^2)\int V(\mathbf{x})\mathscr{J}^0(0, \mathbf{x})d^3x|0\rangle =$$

$$= \lambda\int_0^{M^2}\delta(\mu^2)d\mu^2 = \lambda \tag{II.13}$$

where $E(M^2)$ is the projector on states of mass less or equal to M. Since for $\lambda \neq 0$ (II.13) is different from zero no matter how small M we conclude the existence of discrete eigenstates of the mass operator with eigenvalue zero. q.e.d.

Remarks

1 It is clear that the only zero mass particles of interest are the ones which are coupled to the vacuum by the current. If the current transforms like a four-vector under the Lorentz group, and no use has been made of this fact in our proof, the zero mass states must necessarily have spin 0. (If indefinite metric is allowed they may correspond to the aphysical zero norm part of a spin 1 particle).

2 It is illustrative to compare our arguments with the following naïve proof of Goldstone's theorem.

Consider

$$L(\mathbf{p}, p_0) = \int \langle 0|[\mathscr{J}^0(\mathbf{x}, x_0), A]|0\rangle e^{-i\mathbf{px}+ip_0x_0} \lambda^4 x \qquad \text{(II.14)}$$

Using the continuity equation and dropping boundary terms one concludes as in (II.7)

$$\lim_{\mathbf{p}\to 0} p_0 L(\mathbf{p}, p_0) = 0 \qquad \text{(II.15)}$$

and hence

$$L(\mathbf{0}, p_0) = \lambda\delta(p_0) \qquad \text{(II.16)}$$

This will imply the existence of a discrete excitation whose energy goes to zero with the momentum only if one knows that $L(\mathbf{p}, p_0)$ can be written as $g(\mathbf{p}, p_0 - E(p))$ where g is smooth in its first variable.

This is always the case for relativistic field theories as a consequence of local commutativity since $\rho_{1,2}(p^2, \mathbf{p})$ is the Fourier transform of a function with compact support and therefore analytic in $\mathbf{p}$.

3 Local commutativity is used in two ways in our proof.

First to conclude from the continuity equation that

$$\lim_{R\to\infty} \left\langle 0 \left| \left[\frac{d}{dx_0} \mathscr{J}^0(x_0, f_R), A \right] \right| 0 \right\rangle = 0$$

as in (II.7), secondly to be able to write the representation (II.2). As far as the first result is concerned we could have done with a much weaker form of commutativity. If one assures now a mass gap hypothesis which is stronger than our assumption of no zero mass particles one can put the naïve proof (II.14–16) into rigorous shape [4] to obtain:

$$\lim_{R\to\infty} \langle 0|[\mathscr{J}^0(f_d f_R), \mathscr{P}]|0\rangle = 0 \tag{II.17}$$

where it is sufficient that $|\vec{x}|^2[\mathscr{J}^\mu(x_0, \mathbf{x}), \mathscr{P}] \underset{|\mathbf{x}|\to\infty}{\to} 0$.

This will be of interest in discussing the problem of symmetries in non-relativistic systems.

III Construction of Q and $U(\lambda)$

We are now prepared to show that if there are no zero mass particles in the theory (I.33) defines an operator with all the properties of a charge.

Firstly consistency requires that if

$$A|0\rangle = B|0\rangle \tag{III.1a}$$

with A, B local polynomials

$$QA|0\rangle = QB|0\rangle \tag{III.1b}$$

Proof From (I.33)

$$QA|0\rangle - QB|0\rangle = \underset{R>R_0}{[\mathscr{J}^0(f_d f_R), A-B]}|0\rangle \tag{III.2}$$

Taking the scalar product of both sides of (III.2) with $C|0\rangle$ where C is an arbitrary local polynomial

$$\langle 0|C^+QA|0\rangle - \langle 0|C^+QB|0\rangle = -\underset{R>R_0}{\langle 0|C^+(A-B)\mathscr{J}^0(f_d f_R)|0\rangle}$$

$$= \underset{R>R'_0}{\langle 0|[\mathscr{J}^0(f_d f_R), C^+(A-B)]|0\rangle} \tag{III.3}$$

where use has been made of (III.1), and the fact that $C^+(A-B)$ is a local polynomial.

Using (II.1) and (III.3)

$$\langle 0|C^+QA|0\rangle = \langle 0|C^+QB|0\rangle \tag{III.4}$$

and since $\{C|0\rangle\}$ is a dense set of vectors $QA|0\rangle = QB|0\rangle$ (III.1b). q.e.d.

(Obs. In a local relativistic theory we could have used a simpler proof based on the fact that if $A|0\rangle = B|0\rangle$ then $A = B$. Our proof applies however also to non-relativistic theories.)

Having shown the univocity of the definition (I.33) we show now that the operator Q is hermitian between vectors of the set $\{A|0\rangle\}$.

Proof.

$$\langle 0|B^+QA|0\rangle = \underset{R>R_0}{\langle 0|B^+[\mathscr{J}^0(f_d f_R), A]|0\rangle} = \tag{III.5}$$

$$\underset{R>R'_0}{\langle 0|[B^+, \mathscr{J}^0(f_d f_R]A|0\rangle} + \underset{R>R'_0}{\langle 0|[\mathscr{J}^0(f_d f_R), B^+A]|0\rangle}$$

Using (II.1), the second term on the r.h.s. of (III.5) vanishes and we get

$$\langle 0|B^+QA|0\rangle = \langle 0|A^+QB|0\rangle \qquad \text{q.e.d.} \tag{III.6}$$

The hermitian operator Q defined by (I.33) corresponds to infinitesimal generator of the symmetry associated to a conserved current. To obtain the operator $U(\lambda)$ corresponding to a finite transformation we take the exponential of Q. For internal symmetries the states $A|0\rangle$ are analytic vectors for Q so that $U(\lambda)$ can be directly defined by the convergent power series expansion on the dense set $\{A|0\rangle\}$

$$U(\lambda)A|0\rangle = \sum_{n=1}^{\infty} \frac{(i\lambda Q)^n}{n!} A|0\rangle = A_\lambda|0\rangle \tag{III.7}$$

The convergence of this series is easily proved [11] since for internal symmetries it is reduced to the convergence of a power series of finite dimensional matrices acting on the space of indices.

The hermiticity of Q (III.6) and the convergence (III.7) imply in

$$\langle 0|A^+U^+(\lambda)U(\lambda)B|0\rangle = \langle 0|A^+B|0\rangle \tag{III.8}$$

what allows by continuity to extend $U(\lambda)$ as a unitary operator

$$\langle\psi|U^+(\lambda)U(\lambda)|\Phi\rangle = \langle\psi|\Phi\rangle = \langle\psi|U(\lambda)U^+(\lambda)|\Phi\rangle \tag{III.9}$$

From (III.7) with $A = 1$ one has

$$U(\lambda)|0\rangle = |0\rangle \tag{III.10}$$

and with A replaced by AB

$$U(\lambda)AU^{-1}(\lambda)=A_{\lambda} \tag{III.11}$$

We obtain thus in a constructive way the unitary operator that implements the symmetry. In this respect our approach differs slightly from the one adopted in [4] where the existence of $U(\lambda)$ is shown indirectly via the Gel'fand-Segal construction.

From (III.10, 11), one gets using the Haag-Ruelle collision theory [12], the transformation properties of the asymptotic states, the multiplet structure and the invariance of the S matrix under the symmetry group.

It is clear that our construction only works due to (II.1). In case

$$\langle 0|[\mathscr{J}^0(f_R f_d), A]|0\rangle_{R>R_0} \neq 0 \tag{III.12}$$

a unitary implementation of the symmetry is not possible and one has a *spontaneous symmetry breakdown*.

IV Charges as integrals of densities

We have seen in the previous section how in the absence of zero mass states, one can build the "charge" operator Q.

Let us examine now the meaning of eq. (I.9) that is, in what sense can be the charge described as an integral of the density over the whole space.

First notice that (II.1) can be generalized to read

$$\lim_{R\to\infty} \langle 0|[\mathscr{J}^0(f_d f_R), \mathscr{P}]|0\rangle = 0 \tag{IV.1}$$

with $\mathscr{P}$ a quasi-local polynomial since

$$\|\mathscr{P}|0\rangle - A_R|0\rangle\| \langle \alpha/_R n \tag{IV.2}$$

and

$$\|\mathscr{J}^0(f_d f_R)|0\rangle\| \langle \beta R^3 \tag{IV.3}$$

Now if we make the stronger mass gap assumption it is always possible to obtain for any A with $\langle A\rangle = 0$ a quasi-local polynomial such that

$$\mathscr{P}^+|0\rangle = A^+|0\rangle \tag{IV.4}$$
$$\mathscr{P}|0\rangle = 0$$

and thus

$$\lim_{R\to\infty} \langle 0| A\mathcal{J}^0(f_d f_R)|0\rangle = 0 \qquad \text{(IV.5)}$$

Using now definition (I.33) and eq. (IV.5) one has, with A, B arbitrary local polynomials

$$\langle 0| B^+ QA|0\rangle \underset{R>R_0}{=} \langle 0| B^+[\mathcal{J}^0(f_d f_R), A]|0\rangle = \qquad \text{(IV.6)}$$
$$= \lim_{R\to\infty} \langle 0| B^+[\mathcal{J}^0(f_d f_R), A]|0\rangle = \lim_{R\to\infty} \langle 0| B^+\mathcal{J}^0(f_d f_R)A|0\rangle$$

Equation (IV.6) means that the formal definition of the charge as integral of the density should be understood as

$$\langle \psi| Q|\Phi\rangle \underset{R\to\infty}{=} \langle \psi|\mathcal{J}^0(f_d f_R)|\Phi\rangle \qquad \text{(IV.7)}$$

with $|\psi\rangle$, $|\Phi\rangle$ states obtained by the application of local polynomials on the vacuum. One can extend the validity of eq. (IV.7) to include states obtained by applying quasi-local polynomials on the vacuum and even to states of the form $\int f(\mathbf{x})A(\mathbf{x})^3 x|\infty\rangle$ with $|\mathbf{x}^2| f(\mathbf{x}) \underset{|\mathbf{x}|\to 0}{\to} 0$ [13, 14a]. It is certainly not valid for arbitrary states in the domain of Q as already seen for a free field theory, the physical reason being the appearance of vacuum fluctuations at the surface of the volume over which $\mathcal{J}^0$ is integrated which grow with R^2 and therefore have non-vanishing overlap with the states $|\Phi\rangle$, $|\psi\rangle$ unless those have "wave functions" which tend sufficiently fast to zero for large distances.

Although I can provide no rigorous proof I believe that eq. (IV.7) for quasi-local states is a consequence of (II.1) independent of any additional spectrum assumptions. This is of interest in theories like for instance quantum-electrodynamics where the charge operator exists and the gauge group unitarily implemented even though there are zero mass particles [14a]. Very recently it has been rigorously shown by H. Reeh that (IV.7) holds in quantum-electrodynamics for local states [14b].

In theories with a mass gap (which if they are physically reasonable also imply a gap between the one particle and the continuum states), since one can build a normalized one particle state by applying a quasi-local polynomial on the vacuum [12], one has from (IV.7)

$$\langle 1| Q| 1\rangle = \lim_{R\to\infty} \langle 1|\mathcal{J}^0(f_d f_R)| 1\rangle = F(0) \qquad \text{(IV.8)}$$

where $F(0)$ is the form factor at zero momentum transfer, which is a proof of the well-known statement that coupling constants attached to conserved currents are not renormalized [15].

In quantum-electrodynamics due to the presence of zero-mass particles the canonical current

$$\mathscr{J}_c^\mu = Z_2 \frac{[\bar{\psi}, \gamma^\mu \psi]}{2} \tag{IV.9}$$

which acts as the generator of local gauge transformations

$$[\mathscr{J}_c^\mu(\mathbf{x}), \psi(\mathbf{y})] = -\psi(\mathbf{y})\delta(\mathbf{x}-\mathbf{y}) \tag{IV.10}$$

satisfies

$$\lim_{R\to\infty} \langle 1|\mathscr{J}_c^0(f_d f_R)|1\rangle = Z_3 \neq \langle 1|Q|1\rangle = 1 \tag{IV.11}$$

violating eq. (IV.8).

On the other hand the source current

$$\mathscr{J}_s^\mu = Z_3^{-1}\mathscr{J}_c^\mu \tag{IV.12}$$

$$\Box A^\mu = e\mathscr{J}_c^\mu$$

which satisfies eq. (IV.8) is not the generator of gauge transformations. Those results which are physically understandable as resulting from vacuum polarization effects [14a], are a result of the fact that the canonical current contains a longitudinal part which contributes to the commutator but not to the one-particle matrix elements [5].

As a matter of fact, neither of the currents is a finite operator due to this longitudinal part but one can introduce a finite current which is both the generator and satisfies (IV.8) by

$$\mathscr{J}_F^\mu = \mathscr{J}_s^\mu + \frac{(1-Z_3^{-1})}{e}\partial^\mu(\partial_\nu A^\nu) \tag{IV.13}$$

To end this section it is appropriate to remark that it provides a natural setting for the discussion of Coleman's theorem [16], which we state without proof.

Th. Unitary operators giving rise to approximate symmetries associated with non-conserved currents cannot exist in relativistic field theories.

V Spontaneous symmetry breakdown in many-body systems and difficulties in particle physics

In the previous sections, we have dealt with relativistic theories obeying the postulate of local commutativity. On the other hand since the classical examples of spontaneous symmetry breakdown (as the crystal and the ferromagnet for instance) come from non-relativistic many-body systems it is desirable to extend our results whenever possible to include those theories.

As mentioned in section II (remark 3, e.g. (II.17)) Goldstone's theorem can be generalized from the local framework to theories where the commutators between quasi-local polynomials in the basic fields (I.15) decrease faster than $1/|\vec{x}|^2$ for large spatial separations

$$\lim_{\vec{x}\to\infty} |\vec{x}|^2 \langle 0|[\mathscr{P}^1(\mathbf{x}), \mathscr{P}^2]|0\rangle = 0 \tag{V.1}$$

where $\mathscr{P}^1(\mathbf{x})$ is the translate by $\mathbf{x}$ of the polynomial $\mathscr{P}^1$.

It is clear that in a theory satisfying canonical commutation relations and with a Hamiltonian

$$H = \int \frac{\nabla\psi^+ \nabla\psi}{2m} d^3x + \int \psi^+(\mathbf{x})\psi^+(\mathbf{y})V(\mathbf{x}-\mathbf{y})\psi(\mathbf{x})\psi(\mathbf{y})d^3x d^3y - \mu N \tag{V.2}$$

the behaviour of the commutators should be closely related to the range of the potential V.

As for a free field theory the commutators decrease faster than any inverse power of $|\vec{x}|$ and in the strong coupling limit ($m\to\infty$) one obtains [17]

$$[\psi(\mathbf{x}, t), \psi^+(\mathbf{y}, 0)]_\pm \approx tV(\mathbf{x}-\mathbf{y})$$
$$|\mathbf{x}-\mathbf{y}|\to\infty$$

one can expect that for potentials of sufficiently short range (V.1) is satisfied. (This has been verified up to 2nd order perturbation theory [18].)

One could therefore expect that a general proof of unitary implementation of a symmetry associated to a conserved current could be obtained along the lines and for local relativistic theories for a short range potentials and an energy gap between the ground state and the first excited state. This is however *not* the case due to the fact that there is always one symmetry which is spontaneously broken for any infinite many-body system—*The Galilei invariance* [17,19].

In fact, the expectation value of the matter current changes by going to a reference frame moving with velocity $\mathbf{v}$ with respect to original one by

$$\langle \mathscr{J} \rangle \rightarrow \langle \mathscr{J} \rangle + \mathbf{v} \langle \rho \rangle$$

which clearly shows the spontaneous breakdown of the Galilei symmetry.

From this one concludes using Goldstone's theorem that any such system with short range forces has (phonon-like) excitations of arbitrarily small energy.

A more precise result can be obtained by using the method of sum rules [17].

There is no energy gap for any translationally invariant many-body system with a potential such that $r^{(1+\varepsilon)}_{n\rightarrow\infty} V(r) \rightarrow 0$.

Proof Using the continuity equation

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \mathscr{J} = 0 \tag{V.3}$$

one arrives by a straightforward application of equal-time commutation relations at the following sum-rule [20]

$$\tilde{F}(\mathbf{p}) = \int_0^\infty w \, d\mu_p(w) = \langle \Omega | \rho(0) | \Omega \rangle \mathbf{p}^2 \tag{V.4}$$

where $\tilde{F}(p)$ is the Fourier transform of

$$F(\mathbf{x}) = \left\langle \Omega \left| \left[\frac{\partial \rho}{\partial t}(\mathbf{x}, 0), \rho(0) \right] \right| \Omega \right\rangle \tag{V.5}$$

$|\Omega\rangle$ the ground state, and $d\mu_p(w)$ a positive measure in w.

Taking now the momentum conservation law

$$\frac{\partial}{\partial t} \mathscr{J}_i(\mathbf{x}, t) = \frac{\partial S_{ik}}{\partial x^k} - \psi^+(\mathbf{x}, t) \int \nabla_i V(\mathbf{x}-\mathbf{y}) \rho(\mathbf{y}, t) \psi(\mathbf{x}, t) \tag{V.6}$$

with S_{ik} the stress tensor we find again using equal time commutation relations.

$$\left\langle \Omega \left| \left[\frac{\partial}{\partial t} \mathscr{J}_i(\mathbf{x}, 0), \mathscr{J}_\ell(0) \right] \right| \Omega \right\rangle = \left\langle \Omega \left| \left[\frac{\partial S_{ik}}{\partial x^k}(\mathbf{x}, 0), \mathscr{J}_\ell(0) \right] \right| \Omega \right\rangle + \\ + \frac{\delta^3(\mathbf{x})}{i} \nabla_\ell \int \langle \Omega | \rho(0) \rho(\mathbf{y}, 0) | \Omega \rangle \nabla_i V(\mathbf{x} - y) d^3 y \\ - \frac{1}{i} \langle \Omega | \rho(0) \rho(\mathbf{x}, 0) | \Omega \rangle \nabla_\ell \nabla_i V(\mathbf{x}) \tag{V.7}$$

and using (V.3) once more on the r.h.s. of (V.7) we find

$$\int_0^\infty w^3 d\mu_p(w) = 0(p^2) \text{ if } \lim_{n \to \infty} r^{1+\varepsilon} V(r) = 0 \tag{V.8}$$

Comparing (V.4) and (V.8) one concludes

(i) there is no energy gap

(ii) the weight of $d\mu_p$ gets entirely concentrated at the origin $w = 0$ for $\mathbf{p} \to 0$

$$\lim_{\mathbf{p} \to 0} \frac{\int_{a^2}^\infty d\mu_p(w)}{\int_0^{a^2} d\mu_p(w)} = 0 \tag{V.9}$$

The question whether those Goldstone excitations are of a quasi-particle nature or not cannot however (contrary to the relativistic case) be settled on a general basis and depends on more detailed dynamical information. They are particle like for a free boson system but not for a free fermion gas.

It should be remarked that despite the impossibility of a general theorem on unitary implementation of symmetries like we had in the relativistic case, Goldstone's theorem can still be useful in many-body systems to obtain additional information on the excitation spectrum coming from other broken-symmetries [21].

In particle physics the idea of explaining some of the approximate symmetries observed in nature such as isospin, $SU(3)$, etc. [22], although extremely appealing from the aesthetical point of view has always been plagued by the appearance of unwanted zero-mass particles.

Although one can always hope for, that through some dynamical miracle those zero mass particle do not couple to matter and remain unobservable it seems to me that at least in the case of isospin breaking, since the associated Goldstone bosons are charged particles, they should have an observable effect through macroscopic deviations in the Coulomb law coming from vacuum polarization. A very rough estimate of the lowest order contribution gives a potential which behaves at large distances as $\sim \log r/r$.

A very interesting approach for obtaining spontaneous breakdown without zero mass particles [23] is based upon the introduction of gauge (Yang-Mills) fields which lead to a relativistic theory without local commutation relations so that by a mechanism similar to that played by the Coulomb potential in many-body systems [24] the Goldstone bosons become massive.

Those attempts bring us of course to the very basic question of how much physics is there in the postulate of local-commutativity for non observable fields and under which conditions can this be proved starting from local commutativity for the observables [25].

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